Universal localization at semiprime Goldie ideals

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Motivation

David Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, p. 57:

"A *local ring* is a ring with just one maximal ideal. Every since Krull's paper (1938) local rings have occupied a central position in commutative algebra. The technique of *localization* reduces many problems in commutative algebra to problems about local rings. This often turns out to be extremely useful.

Most of the problems with which commutative algebra has been successful are those that can be reduced to the local case."

It is a major challenge is to try to extend the commutative localization techniques to noncommutative rings. Under the right conditions, things work well. Otherwise, attempts have been made to model the categorical properties of the commutative case, beginning with Gabriel's thesis in 1962. Universal localization offers a more ring-theoretic approach.

Background from commutative algebra

If R is a commutative Noetherian ring, an ideal $P \subseteq R$ is by definition a prime ideal if its complement $R \backslash P$ is a multiplicatively closed set. We can use fractions $\frac{a}{c}$, with $a \in R$, $c \in R \backslash P$, to construct a ring R_P and ring homomorphism $\lambda : R \to R_P$ which inverts the elements of $R \backslash P$. We have the following properties.

(i) The ideal PR_P is the unique maximal ideal of R_P.

(ii) R_P / PR_P is isomorphic to the field of quotients $Q(R/P)$ of R/P.

(iii) For $r \in R$, $\lambda(r) = 0$ iff $cr = 0$ for some $c \in R \backslash P$.

(iv) The functor $R_P \otimes_R - : R{-}\mathrm{Mod} \to R_P{-}\mathrm{Mod}$ takes short exact sequences to short exact sequences, because R_P is flat when considered as a module over R.

Preview of one possible noncommutative definition

$\lambda : R \to R_P$ can be defined as the ring homomorphism universal with respect to the property that if $c \in R \backslash P$ then $\lambda(c)$ is invertible in R_P. That is, if $\phi : R \to T$ inverts $R \backslash P$, then there is a unique ring homomorphism ϕ' such that the following diagram commutes.

$$\begin{tikzcd}
R \arrow[r, "\lambda"] \arrow[dr, "\phi"'] & R_P \arrow[d, dotted, "\phi'"] \\
 & T
\end{tikzcd}$$

$\lambda : R \to R_P$ can also be defined as the ring homomorphism universal with respect to the property that $P = \lambda^{-1}(J(R_P))$ and $R_P/J(R_P) = Q(R/P)$.

The noncommutative case

In a noncommutative ring R an ideal P is called prime if $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$, for all ideals A, B of R.

Example 1. Let $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$ and $P = \begin{bmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{bmatrix}$. Then P is prime since the ideals of R are in one-to-one correspondence with the ideals of the $\mathbb{Z}$.

Note that $R/P \cong \begin{bmatrix} \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \end{bmatrix}$, and that this factor ring has divisors of zero.

The logical candidate for a localization of R at P is $\begin{bmatrix} \mathbb{Z}_{(2)} & \mathbb{Z}_{(2)} \\ \mathbb{Z}_{(2)} & \mathbb{Z}_{(2)} \end{bmatrix}$, which can be constructed by inverting all scalar matrices with an odd entry.

The noncommutative analog of a field of quotients

If R is a subring of Q, then R is a *left order* in Q if
(i) each $c \in R$ that is not a divisor of zero has an inverse in Q, and
(ii) each $q \in Q$ can be written in the form $c^{-1}a$, for $a, c \in R$, where c is regular (not a divisor of zero).

To put the product ac^{-1} into standard form we need to be able to find a_1 and c_1 with $ac^{-1} = c_1^{-1}a_1$, where c_1 is regular. The existence of a_1, c_1 with $c_1 a = a_1 c$ is the *left Ore condition*. Then $c^{-1}a \cdot d^{-1}b$ can be put into standard form by finding a_1 and d_1 with $d_1 a = a_1 d$, so that $ad^{-1} = d_1^{-1}a_1$.

Comment: The Ore condition fails in many (if not most) cases when trying to construct a localization at a prime ideal, so it is a major stumbling block for noncommutative localization.

Analog of a quotient field: Goldie's theorem

Goldie's theorem (1958) shows that R is a left order in a full ring of $n \times n$ matrices over a skew field if and only if R is a prime ring with ACC on left annihilators and finite uniform dimension. (These finiteness conditions always hold when R is left Noetherian.) This ring of quotients is called the *classical ring of left quotients of R* and is denoted by $Q_{cl}(R)$.

We are now ready to look at Cohn's approach to noncommutative localization. We focus on prime ideals of R for which $Q_{cl}(R/P)$ exists. Instead of trying to invert elements in $R \backslash P$, we would like to invert $C(P)$, the set of elements that are regular modulo P. Equivalently, these are the elements inverted by the canonical homomorphism $R \to R/P \to Q_{cl}(R/P)$.

In Example 1, where P is the set of 2×2 matrices with even entries, $C(P)$ is the set of matrices with odd determinant.

Ore localization

If P is a prime Goldie ideal for which $C(P)$ satisfies the left Ore condition and is left reversible (if $ac = 0$ for $c \in C(P)$, then $c'a = 0$ for some $c' \in C(P)$) then the construction of a localization R_P goes through much as in the commutative case, and all four of the properties listed above still hold.

Example 2.

$$R = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}, P_1 = \begin{bmatrix} 2\mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}, P_2 = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & 2\mathbb{Z} \end{bmatrix}.$$

Then $R/P_i \cong \mathbb{Z}/2\mathbb{Z}$ and so $Q_{cl}(R/P_i) = R/P_i$ is a field, making P_i as nice a prime Goldie ideal as possible. But P_1 satisfies the left Ore condition, while P_2 does not.

Checking the Ore condition

The ideal P_1 satisfies the left Ore condition:
given $a = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \in R$ and $c = \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix} \in C(P_1)$ we need to solve $c'a = a'c$ with $c' \in C(P)$.

$$\begin{bmatrix} c_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix}.$$

The ideal P_2 does *not* satisfy the left Ore condition:
given $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in R$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in C(P_2)$ the equation

$$\begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

has no solution with c_{22} odd.

Universal localization

We now turn to properties (i) and (ii) of the commutative case:
(i). $\lambda : R \to R_P$ is universal with respect to the property that if $c \in R \backslash P$ then $\lambda(c)$ has an inverse in R_P.
(ii). The ideal PR_P is the unique maximal ideal of R_P, and R_P/PR_P is isomorphic to $Q(R/P)$.

The ring universal with respect to inverting a set of elements can be defined, but it may be the zero ring. (See Example 9.3 in Lam's book **Lectures on Modules and Rings**).

A new approach inverting matrices rather than elements was introduced by Cohn in **Free Rings and Their Relations** (1971) and *Inversive localization in Noetherian rings*, Commun. Pure Appl. Math. **26** (1973), 679-691.

It's useful to generalize to a semiprime ideal S for which $Q_{cl}(R/S)$ exists and is semisimple Artinian, i.e. for which R/S is a semiprime left Goldie ring. In this case we say that S is a semiprime Goldie ideal.

Definition of the universal localization

Let S be a semiprime Goldie ideal. Then $R \to R/S \to Q_{cl}(R/S)$ inverts all matrices regular modulo S because $Q_{cl}(M_n(R/S)) \cong M_n(Q_{cl}(R/S))$. Let $\Gamma(S) = \cup_{n=1}^{\infty}\Gamma_n(S)$ be the set of all square matrices regular over R/S.

Definition (Cohn, 1973, Noetherian case)

The *universal localization* $R_{\Gamma(S)}$ of R at a semiprime Goldie ideal S is the ring universal with respect to inverting all matrices in $\Gamma(S)$.

That is, if $\phi : R \to T$ inverts all matrices in $\Gamma(S)$, then there exists a unique ϕ' such that the following diagram commutes.

$$\begin{tikzcd}
R \arrow[r, "\lambda"] \arrow[dr, "\phi"'] & R_{\Gamma(S)} \arrow[d, dotted, "\phi'"] \\
 & T
\end{tikzcd}$$

Elementary properties of the universal localization

Note: If S is left localizable, then $R_{\Gamma(S)}$ coincides with the Ore localization R_S defined via elements.

Theorem

Let S be a semiprime Goldie ideal of R.

(a) *(Cohn, 1971) The universal localization of R at S exists.*

(b) *(Cohn, 1971) The canonical mapping $\lambda : R \to R_{\Gamma(S)}$ is an epimorphism in the category of rings.*

(c) *(1981) The ring $R_{\Gamma(S)}$ is flat as a right module over R if and only if S is a left localizable ideal.*

Cohn's construction

Cohn's construction showing that $R_{\Gamma(S)}$ exists:

For each n and each $n \times n$ matrix $[c_{ij}]$ in $\Gamma(S)$,

take a set of n^2 symbols $[d_{ij}]$,

and take a ring presentation of $R_{\Gamma(S)}$ consisting of all of the elements of R, as well as all of the elements d_{ij} as generators;

as defining relations take all of the relations holding in R,

together with all of the relations $[c_{ij}][d_{ij}] = I$ and $[d_{ij}][c_{ij}] = I$ which define all of the inverses of the matrices in $\Gamma(S)$.

Theorem (Cohn, 1971)

Each element of $R_{\Gamma(S)}$ is an entry in a matrix of the form $(\lambda(C))^{-1}$, for some $C \in \Gamma(S)$, where $\lambda : R \to R_{\Gamma(S)}$.

Some information about the kernel

Theorem (1976)

If $K^2 = K$ for a finitely generated ideal $K \subseteq S$, then $K \subseteq \ker(\lambda)$.

Proof.

Let $K = \sum_{i=1}^{n} Rx_i$, for $x_1, \ldots, x_n \in R$. Since $K = K^2$, we have $K = \sum_{i=1}^{n} Kx_i$. For $\mathbf{x} = (x_1, \ldots, x_n)$ we have $\mathbf{x}^t = A\mathbf{x}^t$, where the $n \times n$ matrix A has entries in $K \subseteq S$. Thus $(I_n - A)\mathbf{x}^t = \mathbf{0}^t$. But $I_n - A$ is invertible modulo S, so it certainly belongs to $\Gamma(S)$. Therefore the entries of $\mathbf{x}$ must belong to $\ker(\lambda)$, and so $K \subseteq \ker(\lambda)$. ☐

Corollary

$R_{\Gamma(P)}$ can be determined for a prime ideal P of an hereditary Noetherian prime ring, since in HNP rings each prime ideal is either localizable or idempotent.

Example 2 again

Example 2: $R = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$, $P_2 = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & 2\mathbb{Z} \end{bmatrix}$, $K = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & 0 \end{bmatrix}$.
First approach: P_2 is right localizable, with torsion ideal K and localization $\mathbb{Z}_{(2)}$. This is also the (left) universal localization because of the symmetry.
Second approach: $K^2 = K$, so $K \subseteq \ker\lambda$, for $\lambda : R \to R_{\Gamma(P_2)}$. It follows easily that $K = \ker\lambda$ and $R_{\Gamma(P_2)}$ is isomorphic to $\mathbb{Z}_{(2)}$.
Third approach: Recalling that $P_1 = \begin{bmatrix} 2\mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$, we can invert the scalar matrices in $C(P_1 \cap P_2)$ to obtain $R_{P_1 \cap P_2}$
$= \begin{bmatrix} \mathbb{Z}_{(2)} & 0 \\ \mathbb{Z}_{(2)} & \mathbb{Z}_{(2)} \end{bmatrix}$ with maximal ideal $\widehat{P_2} = \begin{bmatrix} \mathbb{Z}_{(2)} & 0 \\ \mathbb{Z}_{(2)} & 2\mathbb{Z}_{(2)} \end{bmatrix}$.
Factoring out $\cap_{i=n}^{\infty} \widehat{P_2}^{\,n}$ yields $R_{\Gamma(P_2)} \cong \mathbb{Z}_{(2)}$. This illustrates a two-step approach: use the Ore localization at a suitable semiprime ideal, followed by its universal localization at a prime ideal, which in this particular case is just a factor ring.

Some characterizations of $R_{\Gamma(S)}$

Theorem

Let S be a semiprime Goldie ideal of R.

(a) *(Cohn, 1973) $R_{\Gamma(S)}$ modulo its Jacobson radical is naturally isomorphic to $Q_{cl}(R/S)$.*

(b) *(1981) $R_{\Gamma(S)}$ is universal with respect to the property in (a). In fact, $\lambda : R \to R_{\Gamma(S)}$ is characterized by this property.*

Theorem (1981)

Let R be left Noetherian, let N be the prime radical of R, and let $K = \ker(\lambda)$, for $\lambda : R \to R_{\Gamma(N)}$.

(a) *The kernel K is the intersection of all ideals $I \subseteq N$ such that $C(N) \subseteq C(I)$.*

(b) *The ring R/K is a left order in a left Artinian ring, and $R_{\Gamma(N)}$ is naturally isomorphic to $Q_{cl}(R/K)$.*

Symbolic powers of S

Definition

Let S be a semiprime Goldie ideal of R, with $\lambda : R \to R_{\Gamma(S)}$. The n^{th} *symbolic power of* S is $S^{(n)} = \lambda^{-1}(R_{\Gamma(S)}\lambda(S^n)R_{\Gamma(S)})$.

Theorem (1984)

If R is left Noetherian, then the following conditions hold for the symbolic powers of the semiprime ideal S.

(a) *$S^{(n)}$ is the intersection of all ideals I such that $S^n \subseteq I \subseteq S$ and $C(S) \subseteq C(I)$.*

(b) *$C(S)$ is a left Ore set modulo $S^{(n)}$.*

(c) *$R_{\Gamma(S)}\lambda(S^n)R_{\Gamma(S)} = (J(R_{\Gamma(S)})^n$, for all $n > 0$.*

(d) *$R/S^{(n)}$ is an order in the left Artinian ring $R_{\Gamma(S)}/(J(R_{\Gamma(S)}))^n$.*

The connection with Goldie's localization

In two papers in 1967 and 1968, Goldie defined a localization at a prime ideal P of a Noetherian ring R by first factoring out the intersection $\cap_{n=1}^{\infty} P^{(n)}$ of the symbolic powers. He then took the inverse limit of the Artinian quotient rings $Q_{cl}(R/P^{(n)})$, and finally defined an appropriate subring of this inverse limit.

Theorem (Goldie, 1967)

The above localization modulo its Jacobson radical is isomorphic to $Q_{cl}(R/P)$, and modulo the nth power of its Jacobson radical it is isomorphic to the Artinian classical ring of quotients of the ring R modulo the nth symbolic power of P.

Theorem (1984)

Let P be a prime ideal of the Noetherian ring R. Then Goldie's localization of R at P is isomorphic to $R_{\Gamma(P)} / \bigcap_{n=1}^{\infty} J^n$, where J is the Jacobson radical of $R_{\Gamma(P)}$.

Linked prime ideals

Over a commutative Artinian ring R, the injective envelope $E(S)$ of a simple module S has a composition series in which each factor is isomorphic to S. If R is noncommutative, this is no longer true, and nonisomorphic simple modules can be "linked" by appearing in the same composition series of $E(S)$. Over a Noetherian ring, prime ideals P, Q are said to be linked if there exists an ideal A such that $PQ \subseteq A \subset P \cap Q$ and $(P \cap Q)/A$ is torsionfree as both a left R/P-module and as a right R/Q-module.

Theorem (2016, with Christine Leroux)

Let P, Q be prime ideals of a Noetherian ring R. Let R_{PQ} be the universal localization of R/PQ at its prime radical $P \cap Q/PQ$. Then P and Q are linked in R if and only if their extensions in the Artinian ring R_{PQ} are linked, and the canonical linking ideal is the kernel of the mapping $\lambda : R \to R_{PQ}$.

Note that R_{PQ} is an Artinian homomorphic image of the universal localization $R_{\Gamma(P \cap Q)}$ of R at the semiprime ideal $P \cap Q$.

Missing chain conditions on $R_{\Gamma(S)}$

Example 3. If $P_3 = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & 3\mathbb{Z} \end{bmatrix}$, then $R_{\Gamma(P_1 \cap P_3)} = \begin{bmatrix} \mathbb{Z}_{(2)} & 0 \\ \mathbb{Q} & \mathbb{Z}_{(3)} \end{bmatrix}$. This ring is no longer Noetherian. Bill Blair and I had to work much harder to produce such an example for a prime ideal.

Recall that in a ring finitely generated as a module over its center, the *clique* of a prime ideal P is the set of prime ideals with the same intersection down to the center of the ring.

Theorem (2016, with Leroux)

If R is finitely generated as a module over its Noetherian center, and P is a prime ideal that does not contain the intersection of symbolic powers of any other prime ideal in the clique of P, then $R_{\Gamma(P)}$ is the homomorphic image of the Ore localization at the clique of P, and therefore it is Noetherian.

Another construction of the universal localization

Let S be a semiprime Goldie ideal of R. Each element in the universal localization $R_{\Gamma(S)}$ has the form $e_i \lambda(C)^{-1} e_j^t$ for unit vectors $e_i, e_j \in R^n$ and a matrix $C \in \Gamma_n(S)$.

Instead of modeling elements of the form $c^{-1}a$ where $c \in C(S)$, via ordered pairs (c, a), we model elements of the form $\lambda(a)\lambda(C)^{-1}(\lambda(b))^t$ where $C \in \Gamma_n(S)$ and $a, b \in R^n$. This is a modification of the construction given by Macolmson [1982].

Let X be a left R-module. To construct a module of quotients, consider ordered triples (a, C, x^t) where $a \in R^n$, $C \in \Gamma_n(S)$, and $x \in X^n$, for all positive integers n.

If C, U, V are matrices that are already invertible, then $aC^{-1}x^t = aU(VCU)^{-1}Vx^t$. Consequently we say that $(aU, VCU, Vx^t) \equiv (a, C, x^t)$ if U, V are invertible matrices.

Addition of congruence classes

Model for addition: Suppose C, D are already invertible.

$$[a \quad b] \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}^{-1} \begin{bmatrix} x^t \\ y^t \end{bmatrix} = [a \quad b] \begin{bmatrix} C^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} x^t \\ y^t \end{bmatrix} = [aC^{-1} \quad bD^{-1}] \begin{bmatrix} x^t \\ y^t \end{bmatrix} = aC^{-1}x^t + bD^{-1}y^t$$

Definition

$$(a, C, x^t) + (b, D, y^t) = \left([a \quad b], \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}, \begin{bmatrix} x^t \\ y^t \end{bmatrix} \right)$$

This is a commutative, associative binary operation.

Scalar multiplication (without the Ore condition)

Model for scalar multiplication: Suppose C, D are invertible.

$$\begin{bmatrix} a & 0 \end{bmatrix} \begin{bmatrix} C & -r^t b \\ 0 & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ y^t \end{bmatrix} =$$

$$\begin{bmatrix} a & 0 \end{bmatrix} \begin{bmatrix} C^{-1} & C^{-1} r^t b D^{-1} \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ y^t \end{bmatrix} =$$

$$\begin{bmatrix} aC^{-1} & aC^{-1} r^t b D^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ y^t \end{bmatrix} = aC^{-1} r^t \cdot b D^{-1} y^t$$

Definition

$$(a, C, r^t) \cdot (b, D, y^t) = \left(\begin{bmatrix} a & 0 \end{bmatrix}, \begin{bmatrix} C & -r^t b \\ 0 & D \end{bmatrix}, \begin{bmatrix} 0 \\ y^t \end{bmatrix} \right)$$

Constructing a module of quotients

Let K be the subsemigroup generated by all congruence classes of the form $(0, C, x^t)$ and $(a, C, 0^t)$. Then we define $(a, C, x^t) \sim (b, D, y^t)$ if there exist $z_1, z_2 \in K$ with $(a, C, x^t) + z_1 = (b, D, y^t) + z_2$.

The equivalence relation $\sim$ defines a congruence, and modding out by it produces an abelian group.

If C, C_1 are invertible matrices such that $C_1 A = A_1 C$ for matrices A, A_1, then $AC^{-1} = C_1^{-1} A_1$ and so $aAC^{-1}x^t = aC_1^{-1}A_1 x^t$. This motivates the following "left pseudo-Ore condition" which vastly simplifies the proofs.

Lemma

Let $a \in R^m$, $C \in \Gamma_n(S)$, $x \in X^n$, and let A be any $m \times n$ matrix. If there exist $C_1 \in \Gamma_m(S)$ and an $m \times n$ matrix A_1 such that $C_1 A = A_1 C$, then $(aA, C, x^t) \sim (a, C_1, A_1 x^t)$.

The module of quotients and quotient functor

Theorem (1989)

(a) *The above addition and multiplication, modulo the congruence given by* $\sim$*, define a ring of quotients* $\Gamma^{-1}R$ *and a module of quotients* $\Gamma^{-1}X$*.*
(b) *Elements of* $\Gamma^{-1}R$ *are entries in the inverse of a matrix in* $\Gamma(S)$*.*

Theorem (1989)

$\Gamma^{-1}R \cong R_{\Gamma(S)}$ *and* $\Gamma^{-1}X \cong R_{\Gamma(S)} \otimes_R X$*.*

One reason for studying alternate constructions is to try to find useful characterizations of the kernel of $\lambda : R \to R_{\Gamma(S)}$. The equivalence relation for ordered triples can be expressed in terms of the left and right pseudo-Ore conditions, and this yields one characterization of the kernel.

Unfortunately, trying to use the following criteria has proved to be difficult.

Theorem (Gerasimov, 1982)

$r \in \ker(\lambda)$ for $\lambda : R \to R_{\Gamma(S)}$ iff there is a relation of the form

$$\begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ C_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & b_{12} \\ D_{21} & b_{22} \end{bmatrix}$$

with $C_{21}, D_{21} \in \Gamma(S)$.

Theorem (Malcolmson, 1982)

$r \in \ker(\lambda)$ for $\lambda : R \to R_{\Gamma(S)}$ iff there exist $a, b \in R^n$ and $P, Q \in \Gamma_n(S)$ such that

$$r = ab^t \text{ and } (aQ, PQ, Pb^t) = (0, P', c^t) + (d, Q', 0)$$

for some c, d and $P', Q' \in \Gamma(S)$.

Universal localization as a useful language?

Theorem (Forster-Swan, first part)

Let R be commutative Noetherian and ${}_RM$ be finitely generated. The minimal number of generators of M is less than or equal to $\max_{\{P\ prime\}}\{$ the minimal number of generators of M_P $+$ the Krull dimension of $R/P\}$.

The minimal number of generators of M_P can be calculated as the dimension of the vector space $M_P/J(R_P)M_P$ over $R_P/J(R_P)$.

To prove a noncommutative version of the theorem, Stafford had to make a number of adjustments.
Among other “adjustments”, because there was no suitable analog of localization, he replaced $M_P/J(R_P)M_P$ by $Q_{cl}(R/P) \otimes_R M/PM$.

Universal localization as a useful language?

Theorem (with Mauricio Medina)

Let P be a prime ideal of a left Noetherian ring. Then $Q_{cl}(R/P) \otimes_{R/P} M/PM \cong \left(R_{\Gamma(P)} \otimes_R M\right)/J(R_{\Gamma(P)})\left(R_{\Gamma(P)} \otimes_R M\right)$.

Proof: We have the following exact sequences:

$0 \to J(R_{\Gamma(P)}) \to R_{\Gamma(P)} \to Q_{cl}(R/P) \to 0$ as right R-modules,

$0 \to PM \to M \to M/PM \to 0$ as left R-modules.

(1) $Q_{cl}(R/P)$ is a right R/P-module, so it is annihilated by P, and therefore $Q_{cl}(R/P) \otimes_R PM = 0$.

(2) Since M/PM is a left R/P-module, $Q_{cl}(R/P) \otimes_R M/PM = Q_{cl}(R/P) \otimes_{R/P} M/PM$.

(3) The image of the mapping from $J(R_{\Gamma(P)}) \otimes_R M$ into $R_{\Gamma(P)} \otimes_R M$ is $J(R_{\Gamma(P)})\left(R_{\Gamma(P)} \otimes_R M\right)$.

After tensoring

$$\begin{tikzcd}
J(R_{\Gamma(P)}) \otimes_R PM \arrow[r] \arrow[d] & R_{\Gamma(P)} \otimes_R PM \arrow[r] \arrow[d] & Q_{cl}(R/P) \otimes_R PM \arrow[r] \arrow[d] & 0 \arrow[d] \\
J(R_{\Gamma(P)}) \otimes_R M \arrow[r] \arrow[d] & R_{\Gamma(P)} \otimes_R M \arrow[r] \arrow[d] & Q_{cl}(R/P) \otimes_R M \arrow[r] \arrow[d] & 0 \arrow[d] \\
J(R_{\Gamma(P)}) \otimes_R M/PM \arrow[r] \arrow[d] & R_{\Gamma(P)} \otimes_R M/PM \arrow[r] \arrow[d] & Q_{cl}(R/P) \otimes_R M/PM \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 & 0 & 0 & 0
\end{tikzcd}$$

After simplifying

$$\begin{tikzcd}
J(R_{\Gamma(P)}) \otimes_R PM \arrow[r] \arrow[d] & R_{\Gamma(P)} \otimes_R PM \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
J(R_{\Gamma(P)}) \otimes_R M \arrow[r] \arrow[d] & R_{\Gamma(P)} \otimes_R M \arrow[r] \arrow[d] & Q_{cl}(R/P) \otimes_R M \arrow[r] \arrow[d, "\cong"] & 0 \arrow[d] \\
J(R_{\Gamma(P)}) \otimes_R M/PM \arrow[r] \arrow[d] & R_{\Gamma(P)} \otimes_R M/PM \arrow[r] \arrow[d] & Q_{cl}(R/P) \otimes_{R/P} M/PM \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 & 0 & 0 & 0
\end{tikzcd}$$

Conclusion:

$Q_{cl}(R/P) \otimes_{R/P} M/PM \cong \left(R_{\Gamma(P)} \otimes_R M\right) / J(R_{\Gamma(P)}) \left(R_{\Gamma(P)} \otimes_R M\right)$

The language of universal localization makes Stafford's "work-around" look like a method from the theory of commutative localization.

Some questions

1. Is there a better description of the kernel of $R \to R_{\Gamma(P)}$? When is it the intersection of the symbolic powers of P?

2. Can the prime ideals of $R_{\Gamma(S)}$ be characterized?

3. Are there chain onditions weaker than "Noetherian" that might be preserved? e.g. reduced rank on factor rings?

4 How much of the language of commutative localization can be recovered? Perhaps Goldie's localization is to be preferrred?

5. Can more detailed results be obtained for certain classes of rings? e.g. rings finite over their Noetherian center, enveloping algebras, group algebras?

6 Is there an analog of the sheaf of local rings (from the commutative case)?

Thank you!