Graph Inverse Semigroups & Leavitt Path Algebras

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- $\mathbb{M}_n(K)$ is isomorphic to the Leavitt path algebra of the following graph.

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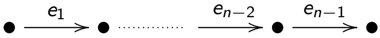

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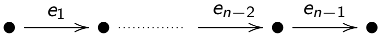

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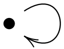
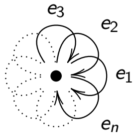


- The Leavitt K-algebra $L_K(n)$ (which is universal with respect to an isomorphism property between finite-rank free modules) is isomorphic to the Leavitt path algebra of the following graph.



Leavitt Path Algebras

Let K be a field and $E = (E^0, E^1, \mathbf{s}, \mathbf{r})$ a (directed) graph (with E^0 the vertex set, E^1 the edge set, and $\mathbf{s}, \mathbf{r} : E^1 \to E^0$ the source and range functions).

The *Leavitt path K-algebra $L_K(E)$ of E* is the K-algebra generated by $E^0 \cup E^1 \cup \{e^{-1} \mid e \in E^1\}$, subject to the following relations.

(V) $vw = \delta_{v,w} v$ for all $v, w \in E^0$.
(E1) $\mathbf{s}(e)e = e\mathbf{r}(e) = e$ for all $e \in E^1$.
(E2) $\mathbf{r}(e)e^{-1} = e^{-1}\mathbf{s}(e) = e^{-1}$ for all $e \in E^1$.
(CK1) $e^{-1}f = \delta_{e,f}\mathbf{r}(e)$ for all $e, f \in E^1$.
(CK2) $v = \sum_{e \in \mathbf{s}^{-1}(v)} ee^{-1}$ for every regular vertex $v \in E^0$.

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Each element of $L_K(E)$ is of the form $\sum_{i=1}^{n} a_i p_i q_i^{-1}$, for some $a_i \in K$ and paths p_i, q_i in E, where $(e_1 \cdots e_n)^{-1} = e_n^{-1} \cdots e_1^{-1}$ for $e_1, \dots, e_n \in E^1$ and $v^{-1} = v$ for $v \in E^0$.

Graph Inverse Semigroup

Given a graph $E = (E^0, E^1, \mathbf{s}, \mathbf{r})$, the *graph inverse semigroup* $\mathcal{S}(E)$ *of* E is the semigroup (with zero) generated by $E^0 \cup E^1 \cup \{e^{-1} \mid e \in E^1\}$, subject to the following relations.

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Each nonzero element of $\mathcal{S}(E)$ is of the form pq^{-1}, for some paths p, q in E, where $(e_1 \cdots e_n)^{-1} = e_n^{-1} \cdots e_1^{-1}$ for $e_1, \ldots, e_n \in E^1$ and $v^{-1} = v$ for $v \in E^0$.

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$\mathcal{S}(E)$ is an inverse semigroup, with $(pq^{-1})^{-1} = qp^{-1}$ for all paths p, q. (A semigroup S is an *inverse semigroup* if for each $x \in S$ there is a unique $y \in S$ satisfying $xyx = x$ and $yxy = y$.)

Path Algebras From Semigroups

Let K be a field and E a graph. The contracted semigroup ring $K[\mathcal{S}(E)]$ (i.e., where the zero of $\mathcal{S}(E)$ is identified with the zero of $K[\mathcal{S}(E)]$) is called the *Cohn path K-algebra of E*.

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The quotient ring

$$K[\mathcal{S}(E)]/\Big\langle v - \sum_{e \in \mathbf{s}^{-1}(v)} ee^{-1} \;\Big|\; v \in E^0 \text{ is regular} \Big\rangle$$

is isomorphic to the Leavitt path algebra $L_K(E)$.

Green's Relations

Given a semigroup S, we denote by S^1 the monoid obtained from S by adjoining an identity element (if S does not already have such an element).

The following relations on elements $x, y \in S$ are known as *Green's relations* (due to James Alexander Green, 1951).

(1) $x \mathscr{L} y$ if and only if $S^1 x = S^1 y$.
(2) $x \mathscr{R} y$ if and only if $xS^1 = yS^1$.
(3) $x \mathscr{J} y$ if and only if $S^1 x S^1 = S^1 y S^1$.
(4) $x \mathscr{H} y$ if and only if $x \mathscr{L} y$ and $x \mathscr{R} y$.
(5) $x \mathscr{D} y$ if and only if $x \mathscr{L} z$ and $z \mathscr{R} y$ for some $z \in S$.

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Question: Which partially ordered sets can be realized as the equivalence classes of a semigroup under a Green's relation?

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Theorem (Ash/Hall, 1975)

Every partially ordered set is order-isomorphic to the set of nonzero $\mathscr{J}$-classes of $\mathcal{S}(E)$, with the usual partial order, for some graph E.

Green's Relations in a Graph Inverse Semigroup

Lemma

Let E be a graph, and let p, q, r, s be paths in E such that $\mathbf{r}(p) = \mathbf{r}(q)$ and $\mathbf{r}(r) = \mathbf{r}(s)$. Then the following hold for $\mathcal{S}(E)$.

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Proof of (1).

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$\iff s = qt$ and $q = su$ for some paths t, u in $E \iff s = q$. $\square$

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Corollary

If E is an acyclic graph, then the nonzero $\mathscr{J}$-classes of $\mathcal{S}(E)$ are in one-to-one correspondence with the vertices of E.

Graph Inverse Semigroup Origins

Theorem (Ash/Hall, 1975)

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Let $(P, \leq)$ be a poset, and let E the graph, where $E^0 = \{v_p \mid p \in P\}$ and $E^1 = \{e_{p,q} \mid p, q \in P \text{ such that } q < p\}$, with $\mathbf{s}(e_{p,q}) = v_p$ and $\mathbf{r}(e_{p,q}) = v_q$.

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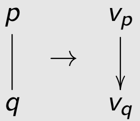

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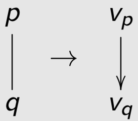
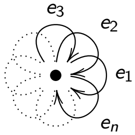


Then E is acyclic, and each nonzero $\mathscr{J}$-class of $\mathcal{S}(E)$ is of the form $[v_p]$ for some $p \in P$. Moreover, for all $p, q \in P$, $q \leq p$ if and only if $\mathbf{r}(t) = v_q$ and $\mathbf{s}(t) = v_p$ for some path t in E, if and only if $[v_q] \subseteq [v_p]$. $\square$

Polycyclic Monoids

Let $n \geq 1$ be an integer and E_n a graph having one vertex and n edges.



Then $\mathcal{S}(E_n)$ is isomorphic to the *polycyclic monoid* (Nivat/Perrot, 1970)

$$P_n = \langle e_1, \ldots, e_n, e_1^{-1}, \ldots, e_n^{-1} \mid e_i^{-1} e_j = \delta_{ij} \rangle,$$

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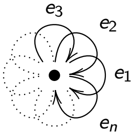

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$\mathcal{S}(E_1) \setminus \{0\}$ is isomorphic to the *bicyclic monoid*

$$P_1 = \langle e, e^{-1} \mid e^{-1} e = 1 \rangle,$$

the canonical inverse semigroup that is not a group.

Posets and Leavitt Path Algebras

For a poset $(P, \leq)$, let E_P be the graph with $E_P^0 = \{v_p \mid p \in P\}$ and

$$E_P^1 = \{e_{p,q}^i \mid i \in \mathbb{N}, \text{ and } p, q \in P \text{ such that } q < p\},$$

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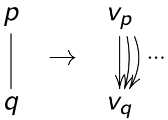

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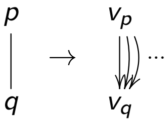

Theorem (Abrams/Aranda Pino/Mesyan/Smith, 2017)

Let $(P, \leq)$ be a poset and K a field. Define $\phi : P \to \mathrm{Spec}(L_K(E_P))$ via

$$\phi(p) = \langle E_P^0 \setminus \{w \in E^0 \mid \exists \text{ path } t \text{ with } \mathbf{s}(t) = w \text{ and } \mathbf{r}(t) = v_p\}\rangle.$$

Then ϕ is an order-embedding, and ϕ is an order-isomorphism if and only if $(P, \leq)$ satisfies DCC. "Many" posets without DCC can be realized as $\mathrm{Spec}(L_K(E_P))$, for some $(P, \leq)$.

Simple Leavitt Path Algebras

Let E be a graph. If $u, v \in E^0$, and there is a path p in E satisfying $\mathbf{s}(p) = u$ and $\mathbf{r}(p) = v$, then we write $u \geq v$.

Let $H \subseteq E^0$. Then H is *hereditary* if whenever $u \in H$ and $u \geq v$ for some $v \in E^0$, then $v \in H$. H is *saturated* if $\mathbf{r}(\mathbf{s}^{-1}(v)) \subseteq H$ implies that $v \in H$ for any regular $v \in E^0$ (i.e., a vertex that emits a nonzero finite number of edges).

Theorem (Abrams/Aranda Pino, 2008)

Let K be a field and E a graph. Then $L_K(E)$ is simple if and only if the following conditions hold.

1. Every cycle in E has an exit.
2. The only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0.

Congruence-Free Graph Inverse Semigroups

Let S be a semigroup. An equivalence relation $R \subseteq S \times S$ is a *congruence* if $(x, y) \in R$ implies that $(xz, yz), (zx, zy) \in R$ for all $x, y, z \in S$. The *diagonal congruence* on S is the relation $\Delta = \{(x, x) \mid x \in S\}$. S is *congruence-free* if its only congruences are $S \times S$ and Δ.

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Theorem (Mesyan/Mitchell, 2016)

Let E be a graph. Then $\mathcal{S}(E)$ is congruence-free if and only if one of the following holds.

1. $|E| = 1$.
2. $|E| \geq 2$, E has only one strongly connected component, and each vertex in E^0 emits at least 2 edges.

Graded Ideals in Leavitt Path Algebras

Let E be a graph. A *breaking vertex* of hereditary and saturated $H \subseteq E^0$ is an infinite emitter $v \in E^0 \backslash H$ such that $0 < |\mathbf{s}^{-1}(v) \cap \mathbf{r}^{-1}(E^0 \backslash H)| < \aleph_0$. The set of all breaking vertices of H is denoted by B_H. For each $v \in B_H$, let

$$v^H := v - \sum_{\mathbf{s}(e)=v,\, \mathbf{r}(e)\notin H} ee^{-1}.$$

An ideal I of $L_K(E)$ is *graded* if $I = I(H, S) = \langle H \cup \{v^H \mid v \in S\}\rangle$ for some hereditary saturated $H \subseteq E^0$ and $S \subseteq B_H$.

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$$(E\backslash(H,S))^1 = \{e \in E^1 \mid \mathbf{r}(e) \notin H\} \cup \{e' \mid e \in E^1 \text{ with } \mathbf{r}(e) \in B_H \backslash S\},$$

and $\mathbf{r}, \mathbf{s}$ are extended to $E\backslash(H,S)$ by setting $\mathbf{s}(e') = \mathbf{s}(e)$ and $\mathbf{r}(e') = \mathbf{r}(e)'$.

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Theorem (Tomforde, 2007)

Let K be a field, E a graph, and $I(H,S)$ a graded ideal of $L_K(E)$. Then $L_K(E)/I(H,S) \cong L_K(E \setminus (H,S))$.

Rees Congruences in Graph Inverse Semigroups

A congruence $R \subseteq S \times S$ on a semigroup S is called a *Rees congruence* if $R = (I \times I) \cup \Delta$ for some ideal I of S.

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Let $E = (E^0, E^1, \mathbf{s}, \mathbf{r})$ be a graph. For $S \subseteq E^0$, we denote by $E \setminus S$ the *quotient* graph $F = (F^0, F^1, \mathbf{r}_F, \mathbf{s}_F)$, where $F^0 = E^0 \setminus S$,

$$F^1 = E^1 \setminus \{e \in E^1 \mid \mathbf{s}(e) \in S \text{ or } \mathbf{r}(e) \in S\},$$

and $\mathbf{r}_F$, $\mathbf{s}_F$ are the restrictions of $\mathbf{r}$, $\mathbf{s}$, respectively, to F^1.

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Theorem (Mesyan/Mitchell, 2016)

Let E be a graph, R a Rees congruence on $\mathcal{S}(E)$, and I the ideal of $\mathcal{S}(E)$ corresponding to R. Then $\mathcal{S}(E)/R \cong \mathcal{S}(E \setminus (I \cap E^0))$.

Rees Congruences in Graph Inverse Semigroups

Proof.

Let $F = E \setminus (I \cap E^0)$. Define $\varphi : \mathcal{S}(E) \to \mathcal{S}(F)$ by

$$\varphi(pq^{-1}) = \begin{cases} pq^{-1} & \text{if } p, q \text{ are paths in F} \\ 0 & \text{otherwise} \end{cases}$$

for all paths p, q in E, and $\varphi(0) = 0$.

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To show that φ is a homomorphism, compare $\varphi(\mu)\varphi(\nu)$ to $\varphi(\mu\nu)$ separately in each of the following cases: (1) $\mu \in I$ or $\nu \in I$; (2) $\mu, \nu \notin I$ and $\mu\nu = 0$; (3) $\mu, \nu \notin I$ and $\mu\nu \neq 0$.

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If $pq^{-1} \in I$ for some paths p, q in E with $\mathbf{r}(p) = \mathbf{r}(q)$, then $\mathbf{r}(p) \in I$, by the classification of the $\mathscr{J}$-classes of $\mathcal{S}(E)$, and hence p, q are not paths in F. It follows that $\varphi(\mu) = 0$ if and only if $\mu \in I$, for all $\mu \in \mathcal{S}(E)$, and hence $R = \ker(\varphi)$.

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Since φ is clearly surjective, $\mathcal{S}(E)/R \cong \mathcal{S}(F)$, by the first isomorphism theorem for semigroups. □

Arbitrary Ideals in Leavitt Path Algebras

Theorem (Rangaswamy, 2014)

Let K be a field, E a graph, and I an ideal of $L_K(E)$, with $H = I \cap E^0$ and $S = \{v \in B_H \mid v^H \in I\}$. Then

$$I = I(H, S) + \sum_{i \in Y} \langle f_i(c_i) \rangle,$$

where Y is a possibly empty index set; each c_i is a cycle without exits in $E \backslash (H, S)$; and each $f_i(x) \in K[x]$ is a polynomial with a nonzero constant term, which is of smallest degree such that $f_i(c_i) \in I$.

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Theorem (Abrams/Ara/Siles Molina, 2017)

Let K be a field and E a graph. Then the lattice of ideals of $L_K(E)$ (under $+$ and $\cap$) is order-isomorphic to the lattice of ordered triples $((H, S), C, P)$, where $H \subseteq E^0$ is hereditary and saturated, $S \subseteq B_H$, C is a set of cycles without exits in $E \backslash (H, S)$, and $P \subseteq K[x]$ is a set of polynomials with a nonzero constant term (with appropriately defined lattice operations).

Arbitrary Congruences in Graph Inverse Semigroups

Let E be a graph. Given $H \subseteq E^0$, denote by $C(H)$ the set of all cycles $c = e_1 \dots e_n$ in E such that $\mathbf{s}(e_i) \in H$ for each i. A *congruence triple* (H, W, f) on E consists of a hereditary set $H \subseteq E^0$, a set

$$W \subseteq \{v \in E^0 \setminus H \mid |\mathbf{s}^{-1}_{E \setminus H}(v)| = 1\},$$

and a *cycle* function $f : C(E^0) \to \mathbb{Z}^+ \cup \{\infty\}$ (i.e., $f(c) = 1$ for all $c \in C(H)$, $f(c) = \infty$ for all $c \notin C(H \cup W)$, and the restriction of f to $C(W)$ is invariant under cyclic permutations). Given a congruence triple (H, W, f), let $\varrho(H, W, f)$ be the congruence on $\mathcal{S}(E)$, generated by

$$(H \times \{0\}) \cup \{(w, ee^{-1}) \mid w \in W,\ \mathbf{s}(e) = w,\ \mathbf{r}(e) \notin H\}$$
$$\cup \{(c^{f(c)}, \mathbf{s}(c)) \mid c \in C(W),\ f(c) \in \mathbb{Z}^+\}.$$

Theorem (Luo/Wang, 2021)

Let E be a graph. Then $(H, W, f) \mapsto \varrho(H, W, f)$ is an order-isomorphism from the lattice of all congruence triples on E (with appropriately defined lattice operations) to the lattice of congruences on $\mathcal{S}(E)$.

Congruence Lattices of Graph Inverse Semigroups

A partially ordered set $(L, \leq)$ is a *lattice*, if for all $a, b \in L$ there exists an greatest lower bound (*meet*) $a \wedge b$ and a least upper bound (*join*) $a \vee b$.

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Let L be a lattice, and $a, b \in L$. We say that b *covers* a, and write $a \prec b$, if $a < b$ and there is no $c \in L$ such that $a < c < b$.

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L is *modular* if $a \leq c$ implies that $(a \vee b) \wedge c = a \vee (b \wedge c)$, for all $a, b, c \in L$. Every modular lattice is both upper- and lower-semimodular, and the converse also holds for finite lattices.

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Theorem (Luo/Wang, 2021)

For any graph E, the lattice of congruences on $\mathcal{S}(E)$ is upper-semimodular.

Congruence Lattices of Graph Inverse Semigroups

Let E be a graph and $v \in E^0$. We refer to v as a *forked* vertex, if there exist distinct edges $e, f \in \mathbf{s}^{-1}(v)$ such that the following properties hold.

(1) $\mathbf{r}(g) \not\geq \mathbf{r}(e)$ for all $g \in \mathbf{s}^{-1}(v) \setminus \{e\}$.
(2) $\mathbf{r}(g) \not\geq \mathbf{r}(f)$ for all $g \in \mathbf{s}^{-1}(v) \setminus \{f\}$.

Theorem (Anagnostopoulou-Merkouri/Mesyan/Mitchell, 2024)

For any graph E, the lattice of congruences on $\mathcal{S}(E)$ is lower-semimodular if and only if E has no forked vertices.

Atoms

Let L be a lattice with a least element 0. We say that $a \in L$ is an *atom* if $0 \prec a$. Also, L is *atomistic* if every element of L can be expressed as a join of finitely many atoms.

Theorem (Anagnostopoulou-Merkouri/Mesyan/Mitchell, 2024)

Let E be a graph. Then every congruence in $\mathcal{S}(E)$ is the join of a (possibly infinite) collection of atoms if and only if for every $v \in E^0$ one of the following holds.

1. $|\mathbf{s}^{-1}(v)| = 0$.
2. $|\mathbf{s}^{-1}(v)| = 1$, v does not belong to a cycle, and $v > u$ for some $u \in E^0$ such that $|\mathbf{s}^{-1}(u)| \neq 1$.
3. $|\mathbf{s}^{-1}(v)| \geq 2$, and $\mathbf{r}(e) \geq v$ for all $e \in \mathbf{s}^{-1}(v)$.

Moreover, the lattice of congruences on $\mathcal{S}(E)$ is atomistic if and only if, in addition to the above conditions on all vertices, E^0 has only finitely many strongly connected components and vertices v such that $|\mathbf{s}^{-1}(v)| = 1$.

Topologies on the Bicyclic Monoid

If S is a semigroup and $\mathcal{O}$ is a topology on S, then we say that S is a *topological semigroup* with respect to $\mathcal{O}$, or that $\mathcal{O}$ is a *semigroup topology* on S, if the multiplication operation $* : S \times S \to S$ on S is continuous with respect to $\mathcal{O}$, where $S \times S$ is endowed with the product topology.

Theorem (Eberhart/Selden, 1969)

Let $P_1 = \mathcal{S}(E_1) \setminus \{0\}$ denote the bicyclic monoid.

1. The only Hausdorff topology on P_1 which makes into a topological semigroup is the discrete topology.
2. If P_1 is contained densely in a Hausdorff topological semigroup S, then P_1 is open in S and $S \setminus P_1$ is an ideal of S, provided it is nonempty.
3. Let $\overline{P_1}$ be the closure of P_1 in a Hausdorff topological inverse semigroup. Then $\overline{P_1} \setminus P_1$ is a group that contains a dense cyclic subgroup, provided it is nonempty.

Topologies on Graph Inverse Semigroups

Theorem (Eberhart/Selden, 1969)

Let $P_1 = \mathcal{S}(E_1) \setminus \{0\}$ denote the bycyclic monoid.

1. The only Hausdorff topology on P_1 which makes into a topological semigroup is the discrete topology.
2. If P_1 is contained densely in a Hausdorff topological semigroup S, then P_1 is open in S and $S \setminus P_1$ is an ideal of S, provided it is nonempty.

Theorem (Mesyan/Mitchell/Morayne/Péresse, 2016)

Let E be a graph.

1. If $\mathcal{S}(E)$ is a topological semigroup with respect to a Hausdorff topology, then $\mathcal{S}(E) \setminus \{0\}$ must be discrete in this topology.
2. If $\mathcal{S}(E)$ is contained densely in a Hausdorff topological semigroup S, then $\mathcal{S}(E) \setminus \{0\}$ is open in S and $(S \setminus \mathcal{S}(E)) \cup \{0\}$ is an ideal of S.

Non-Discrete Semigroup Topology

Proposition (Mesyan/Mitchell/Morayne/Péresse, 2016)

Let E be a graph having paths of arbitrary (finite) length, define $d'(0,0) = 0$,

$$d'(xy^{-1}, 0) = d'(0, xy^{-1}) = \frac{1}{\min\{|x|, |y|\} + 1}$$

for all paths x, y in E, and extend d' to a map $d : \mathcal{S}(E) \times \mathcal{S}(E) \to \mathbb{R}$ via

$$d(\mu, \nu) = \begin{cases} d'(\mu, 0) + d'(\nu, 0) & \text{if } \mu \neq \nu \\ 0 & \text{if } \mu = \nu \end{cases}.$$

Then d is a metric that induces a non-discrete semigroup topology on $\mathcal{S}(E)$.

Complements in Closures

Theorem (Eberhart/Selden, 1969)

Let $P_1 = \mathcal{S}(E_1) \setminus \{0\}$ denote the bicyclic monoid, and let $\overline{P_1}$ be the closure of P_1 in a Hausdorff topological inverse semigroup. Then $\overline{P_1} \setminus P_1$ is a group that contains a dense cyclic subgroup, provided it is nonempty.

Theorem (Mesyan/Mitchell/Morayne/Péresse, 2016)

Let E be a graph, let $\overline{\mathcal{S}(E)}$ be the closure of $\mathcal{S}(E)$ in a Hausdorff topological inverse semigroup, and set $T = \overline{\mathcal{S}(E)} \setminus \mathcal{S}(E)$.

1. For all $\rho \in T$ there are idempotents $\mu, \nu \in T$ such that $\rho \in \mu T \nu$.
2. For all idempotents $\mu, \nu \in T$, if $\mu \neq \nu$, then $\mu\nu = 0$.
3. For any idempotent $\mu \in T$, the set $\mu T \mu \setminus \{0\}$ is a group with identity μ.
4. For any idempotent $\mu \in T$, the group $\mu T \mu \setminus \{0\}$ contains a dense cyclic subgroup.

Complements of Polycyclic Monoids

Proposition (Mesyan/Mitchell/Morayne/Péresse, 2016)

Let $n \geq 2$ be an integer, and suppose that the polycyclic monoid

$$P_n = \langle e_1, \ldots, e_n, e_1^{-1}, \ldots, e_n^{-1} \mid e_i^{-1} e_j = \delta_{ij} \rangle$$

is a subsemigroup of a Hausdorff topological semigroup. Then $\overline{P_n} \setminus P_n$ is either empty or infinite.

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Example (Mesyan/Mitchell/Morayne/Péresse, 2016)

Let S be the monoid with zero element generated by $\{e_1, e_2, e_1^{-1}, e_2^{-1}, X\}$, subject to the relations:

$$e_i^{-1} e_j = \delta_{ij}, \ e_1 X = X e_2^{-1} = X, \ e_1^{-1} X = X e_2 = 0.$$

Then there is a semigroup Hausdorff topology on S, with respect to which S contains a dense non-discrete copy of P_2 (namely, $\langle e_1, e_2, e_1^{-1}, e_2^{-1} \rangle$), and $|S \setminus P_2| = \aleph_0$.

Uncountable Complement Example

Let $L = \{(p_1, p_2, \ldots) \mid p_i \in \{e_1, e_2\}\}$, $R = \{(\ldots, q_2^{-1}, q_1^{-1}) \mid q_i \in \{e_1, e_2\}\}$, where e_1, e_2 are the generators of P_2 as an inverse semigroup.

Let $S \subseteq L \times R$ be the set of pairs $((p_1, p_2, \ldots), (\ldots, q_2^{-1}, q_1^{-1}))$ such that

$$\liminf_{n\to\infty} \frac{|\{1 \le i \le n \mid p_i = e_1\}|}{n} > \frac{1}{2}$$

and

$$\limsup_{n\to\infty} \frac{|\{1 \le i \le n \mid q_i^{-1} = e_1^{-1}\}|}{n} < \frac{1}{2}.$$

Let $T = P_2 \cup S$. Then $|T \setminus P_2| = 2^{\aleph_0}$, and T can be made into a topological semigroup such that $\overline{P_2} = T$.

Details

For the sake of brevity, we denote an element $((p_1, p_2, \ldots), (\ldots, q_2^{-1}, q_1^{-1}))$ of S by $p_1 p_2 \cdots q_2^{-1} q_1^{-1}$.

Define multiplication on T so that it extends the usual multiplication on P_2, where $\sigma\tau = 0$ for all $\sigma, \tau \in S$, and where for all $x \in \{e_1, e_2\}$ and $\sigma = p_1 p_2 \cdots q_2^{-1} q_1^{-1} \in S$,

$$\begin{aligned}
x \cdot \sigma &= x p_1 p_2 \cdots q_2^{-1} q_1^{-1} \\
\sigma \cdot x^{-1} &= p_1 p_2 \cdots q_2^{-1} q_1^{-1} x^{-1} \\
\sigma \cdot x &= \begin{cases} p_1 p_2 \cdots q_2^{-1} & \text{if } x = q_1 \\ 0 & \text{if } x \neq q_1 \end{cases} \\
x^{-1} \cdot \sigma &= \begin{cases} p_2 \cdots q_2^{-1} q_1^{-1} & \text{if } x = p_1 \\ 0 & \text{if } x \neq p_1. \end{cases}
\end{aligned}$$

Details

Identify each $p_1 \dots p_n q_m^{-1} \dots q_1^{-1} \in P_2$ $(p_i, q_i \in \{e_1, e_2\})$ with

$$p_1, \dots, p_n, 1, 1, \dots, 1^{-1}, 1^{-1}, q_m^{-1}, \dots, q_1^{-1}.$$

Define $d : T \times T \to \mathbb{R}$ by

$$d(\sigma, \tau) = \begin{cases} 0 & \text{if } \sigma = \tau \\ \dfrac{1}{\min\{i \mid p_i \neq x_i \text{ or } q_i \neq y_i\}} & \text{if } \sigma \neq \tau, \end{cases}$$

where $\sigma = p_1 p_2 \cdots q_2^{-1} q_1^{-1}$ and $\tau = x_1 x_2 \cdots y_2^{-1} y_1^{-1}$.

Then d is a metric, which induces a topology on $T = P_2 \cup S$ with the desired properties.

Isomorphism and Equivalence Problems

Question (Abrams/Ánh/Louly/Pardo, 2008)

If E and F are finite graphs such that $L_K(E)$ and $L_K(F)$ are purely infinite simple, and $K_0(L_K(E)) \cong K_0(L_K(F))$ via an isomorphism which takes $[1_{L_K(E)}]$ to $[1_{L_K(F)}]$, are $L_K(E)$ and $L_K(F)$ necessarily isomorphic?

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Conjecture (Hazrat, 2013)

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Theorem (Krieger, 2006; Costa/Steinberg, 2016)

The following are equivalent for all graphs E_1 and E_2.

1. $E_1 \cong E_2$.
2. $\mathcal{S}(E_1) \cong \mathcal{S}(E_2)$.
3. $\mathcal{S}(E_1)$ and $\mathcal{S}(E_2)$ are Morita equivalent.

Thank you!